$$(3)\qquad c_n p^n + c_{n-1} p^{n-1} + \ldots + c_1 p + c_0 + \frac{c_n + c_{n-1} + \ldots + c_1 + c_0}{p-2}$$
$$= r_v p^v + r_{v-1} p^{v-1} + \ldots + r_1 p + r_0 + \frac{n}{p-2},$$

from which the values of the c's (and hence by (1) the value or values of m) may be found.

As an example, take the following case: $p=5$ and $s=3725$. We have from (3),

$$c_n p^n + c_{n-1} p^{n-1} + \ldots + c_1 p + c_0 + \frac{c_n + c_{n-1} + \ldots + c_1 + c_0}{p-2}$$
$$= 5^5 + 2.5^4 + 4.5^3 + 3.5^2 + 3.5 + 1 + \tfrac{2}{3}.$$

It is easy to show that $n=5$ and $c_5=1$; then that $c_4=2$, and $c_3=4$, and $c_2=3$, and $c_1=2$, and $c_0=2$. (Notice that the values of the c's are most easily determined in the order given.) This gives

$$m = 5^5 + 2.5^4 + 4.5^3 + 3.5^2 + 2.5 + 2 = 4962.$$

It is now evident that the solution or solutions may *always* be readily obtained from (3).

When $s=5$ and $p=3$, $m=7$ or 9; and this is sufficient to show that more than one solution exists in some cases even when p is odd.

An incomplete solution was received from Professor Zerr.

143. Proposed by JOHN D. WILLIAMS (being the first of 14 challenge problems published in 1832).

Make $x^2+y^2=a^2=z^2+w^2$ and $x^2-w^2=z^2-y^2=\Box$.

I. Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

Let $x=2q(7q-5p)$, $y=25p^2-70pq+48q^2$,
$z=24p^2-70pq+50q^2$, $w=p(10q-7p)$.
$\therefore x^2+y^2=z^2+w^2=(25p^2-70pq+50q^2)^2=a^2$.
$x^2-w^2=z^2-y^2=196q^4-280pq^3+140p^3q-49p^4=(14q^2-10pq-\tfrac{25}{7}p^2)^2$
when $q=\dfrac{1513p}{1680}$; $m=\tfrac{1513}{1680}$, where $pm=q$.

$$\therefore x=\frac{473569p^2}{201600},\ y=\frac{52319p^2}{58800},\ z=\frac{85345p^2}{56448},\ w=\frac{337p^2}{168}.$$

Reducing to a common denominator and expunging common factors, we get

$x=3314983p^2$, $y=1255656p^2$, $z=2133625p^2$, $w=2830800p^2$.
$\therefore x^2+y^2=z^2+w^2=a^2=(3544825p^2)^2$,
$x^2-w^2=z^2-y^2=b^2=(1725017p^2)^2$.
Also $a-x=2(339p)^2=2c^2$, $a-z=2(840p)^2=2d^2$,
$a-y=(1513p)^2=h^2$, $a-w=(845p)^2=k^2$.

II. Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philahelphia, Pa.

General expressions for the sides of duplicate right-triangles having the same hypotenuse are

$x=(p^2-q^2)(r^2-s^2)+4pqrs,\ y=2rs(p^2-q^2)-2pq(r^2-s^2),$

$z=(r^2-s^2)(p^2+q^2),\ w=2rs(p^2+q^2).$

$\therefore x^2+y^2=z^2+w^2=[(r^2+s^2)(p^2+q^2)]^2$, Let $p=2,\ q=1$.

$\therefore x=3(r^2-s^2)+8rs,\ y=6rs-4(r^2-s^2),\ z=5(r^2-s^2),\ w=10rs.$

$x^2+y^2=z^2+w^2=[5(r^2+s^2)]^2,\ x^2-w^2=z^2-y^2=9r^4+48r^3s-54r^2s^2-48rs^3+9s^4=\square$. This is a square when $r=\frac{17}{12}s$.

$$\therefore x=\frac{689s^2}{64},\ y=\frac{161s^2}{36},\ z=\frac{725s^2}{144},\ w=\frac{85s^2}{6}.$$

$\therefore x=2067s^2,\ y=644s^2,\ z=725s^2,\ w=2040s^2.$

$x^2+y^2=z^2+w^2=(2165s^2)^2=a^2,\ x^2-w^2=z^2-y^2=(333s^2)^2=b^2.$

A solution of this problem is given in J. D. Williams' *Algebra*, page 419. He starts with $a^2=b^2+f^2=c^2+e^2$, and $b^2-c^2=d^2=e^2-f^2$. Then he is to make a^2-b^2 a square, a^2-c^2 a square, and b^2-f^2 a square. He assumes $a^2=(p^2+q^2)(r^2+s^2)$, $b=pr\pm qs$, $c=ps\pm qr$. Then he assumes $r=pm-qn$, $s=pn+qm$. He finally arrives at the conclusion that $a=697$, $b=680$, $f=153$, $c=672$, $e=185$, a set of erroneous values, as Dr. Zerr has pointed out. It is likely that Williams' solution may be carried out so that a set of correct values may be obtained. Williams proposed this problem in 1832 as a challenge problem to the mathematicians of the United States. ED. F.

144. Proposed by JOHN D. WILLIAMS (being the ninth of his 14 challenge problems proposed in 1832).

Make $(m^2+n^2)^2x^2\pm(m^2+n^2)x=\square$, $(m^2-n^2)^2x^2\pm(m^2-n^2)x=\square$, and $4m^2n^2x^2\pm2mnx=\square$.

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

Let $m^2+n^2=p,\ m^2-n^2=q,\ 2mn=r.$

$\therefore p^2x^2\pm px=\square,\ q^2x^2\pm qx=\square,\ r^2x^2\pm rx=\square\ldots(1,\ 2,\ 3).$

Let $p^2x^2+px=a^2x^2;\ \therefore x=p/(a^2-p^2).$

This value of x in (2) and (3) gives

$$q^2p^2+qp(a^2-p^2)=\square,\quad r^2p^2+rp(a^2-p^2)=\square\ldots(4,\ 5).$$

Let $q^2p^2+qp(a^2-p^2)=[pq-b(a-p)]^2.$

$\therefore (a-p)=\dfrac{2pq(b+p)}{b^2-pq}$. This value of a in (5) gives

$$r^2b^4+4b^3pqr+2b^2pqr(2p+2q-r)+4bp^2q^2r+p^2q^2r^2=\square$$
$$=(rb+2bpq-pqr)^2,\text{ suppose.}$$

$$\therefore b=\frac{2pqr}{pr+qr-pq};\ x=\frac{p}{a^2-p^2}=\frac{(b^2-pq)^2}{4bpq\ (b+p)\ (b+q)}.$$

$$\therefore x=\frac{-[(pr+qr-pq)^2-4pqr^2]^2}{8pqr(pr+qr-pq)(pq-pr+qr)(pq+pr-qr)}.$$